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LETTER TO THE EDITOR

Universality in a generalised Potts model

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Abstract. A three-state Potts model with pure three-site interactions is analysed using low-temperature series expansions. Universality predicts that the exponents α' and β should be the same as for the 'hard-hexagons' lattice gas: $\frac{1}{3}$ and $\frac{1}{9}$ respectively. The series estimates are consistent with these predictions, and the estimates of γ' are consistent with the scaling value of $1\frac{4}{5}$.

After being of purely theoretical interest for several decades, the Potts models have recently been considered as realistic models of transitions in adsorbed gas monolayers. At the heart of any attempt to apply any lattice model to a real system is the belief in universality—the theory that only the basic symmetries of the interactions are important in determining the type of transition and that other details of the interactions are irrelevant. While universality has been extensively tested in the n-vector models, it has not been possible to determine whether exponents differ between various types of Potts model because of the difficulties in determining exponents in even a single case. For $q=3$, series estimates for the exponent α have ranged from $\alpha = 0{\cdot}05 \pm 0{\cdot}10$ (Straley and Fisher 1973) to $\alpha = 0{\cdot}42 \pm 0{\cdot}05$ (de Neef and Enting 1977). Similarly, renormalisation group techniques have given a wide range of exponent estimates and, prior to the work of Nienhuis *et al* (1979), there had not been a calculation that correctly predicted the change from a continuous transition to a first-order transition as the number of states increased.

Baxter *et al* (1978) have described a generalised triangular lattice Potts model with both two-site and three-site interactions. Enting (1978a) analysed series for this system for various values of the interaction strengths (for the three-state case) and concluded that the variation in the exponent estimates was less than the (rather considerable) uncertainties in the values for the Potts models with pure two-site interactions.

The present work analyses the three-site Potts model with pure three-site interactions. There are a number of reasons for investigating this particular system:

(i) It extends the range of parameters considered by Enting (1978a).

(ii) Since the system is self-dual, it is possible to find the exact solution for the critical energy rather than for the critical value of an 'energy-like' function as considered by Enting.

(iii) New series expansion techniques enable one to calculate series that are longer than was previously possible.

(iv) Similarly the new series expansion techniques have led to more reliable estimates of the exponents for the pure two-site model (Enting 1980a) with which the present results must be compared.

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(v) Most importantly, Baxter (1980) has recently solved the problem of the 'hard-hexagons' lattice gas. This system should lie in the universality class of the three-state Potts model (Alexander 1975). The present work can be regarded as checking that such a three-state Potts universality class does in fact exist.

The system to be analysed is defined on a triangular lattice. At each lattice site is a spin variable that can be in one of three states 0, 1 or 2. The interaction energy is defined on only triangles of 'up' parity. It is 0 if all three states are in the same state and $+J>0$ otherwise. All '1' and '2' sites have a field energy H.

Low-temperature series were calculated using the finite-lattice method, essentially as described by Enting (1978b). The triangular lattice is treated as a square lattice with diagonal bonds. The partition function was obtained by combining partition functions of finite rectangles (m sites by n sites, $m+n \leqslant 17$) using fixed (state '0') boundaries. The main new feature was that, for reasons of computational efficiency, the rectangles were built up one site at a time (as described by Enting 1980b) rather than one column at a time.

The series expansion variable is $u=\exp(-J/kT)$. The coefficients of each power of u are conventionally written as polynomials in $\mu=\exp(-H/kT)$, but they can equally well be written as polynomials in $x=1-\mu$. Furthermore, the calculation can be carried out in a consistent manner keeping only x^0, x^1 and x^2 terms at each stage, giving a considerable reduction in computation.

The series given in table 1 are for the partition function Z, the order parameter M and the zero-field susceptibility χ, in powers of u through u^{18}.

$$M=1-\frac{3}{2}\frac{\partial Z}{\partial x}\Big/Z, \tag{1}$$

$$\chi=\frac{\partial^2 Z}{\partial x^2}\Big/Z-\frac{\partial Z}{\partial x}\Big/Z-\left(\frac{\partial Z}{\partial x}\Big/Z\right)^2, \tag{2}$$

and the energy per site, E, is given by

$$\frac{E}{J}=u\frac{\partial Z}{\partial u}\Big/Z. \tag{3}$$

Using duality, Baxter *et al* (1978) predicted that the critical point should be at the self-dual point,

$$u_c=\tfrac{1}{4}, \tag{4}$$

and that the critical value of E/J should be $\frac{1}{3}$.

The series analysis of Enting (1978a) had confirmed that the duality relation appeared to predict correctly the location of the critical point in the mixed two-site/three-site system. This has subsequently been confirmed by Wu and Zia (1981) on the basis of careful consideration of the topology of the phase diagram.

The series for the pure three-site system were analysed using Padé approximants. Approximants to $(\mathrm{d}/\mathrm{d}u)\ln M$ consistently showed singularities at $u=0{\cdot}249\pm0{\cdot}001$ and $u=-0{\cdot}265\pm0{\cdot}005$. The value of the physical root confirmed the predictions of duality, and all subsequent analysis assumes $u_c=\frac{1}{4}$.

Table 2 lists the values (at $u=\frac{1}{4}$) of Padé approximants to (i) $(u_c-u)(\mathrm{d}/\mathrm{d}u)\ln[(E_c-E)/J]$ (i.e. estimates of $\alpha'-1$), (ii) $(u_c-u)(\mathrm{d}/\mathrm{d}u)\ln M$ (i.e. estimates of $-\beta$) and (iii) $(u_c-u)(\mathrm{d}/\mathrm{d}u)\ln(\chi/u^3)$ (i.e. estimates of γ').

Table 1. Coefficients in low-temperature series expansions. $Z = \Sigma_n a_n u^n$, $M = \Sigma_n b_n u^n$, $\chi = \Sigma c_n u^n$.

n	a_n	b_n	c_n
0	1	1	0
1	0	0	0
2	0	0	0
3	2	−3	2
4	0	0	0
5	12	−36	48
6	−2	−3	34
7	90	−423	894
8	−30	−81	1 086
9	788	−5 157	15 502
10	−366	−1 629	24 954
11	7 626	−64 989	261 510
12	−4 144	−30 234	508 144
13	78 984	−840 096	4 345 152
14	−45 696	−531 279	9 699 474
15	859 212	−11 087 595	71 628 010
16	−494 892	−9 021 717	178 013 490
17	9 696 984	−148 832 739	1 175 190 774
18	−5 258 816	−149 253 585	3 181 358 414

Table 2. Exponent estimates from $[N/D]$ approximants to $(u_c - u)f(u)$ for thermodynamic functions: (i) estimates of $\alpha' - 1$; (ii) estimates of $-\beta$; (iii) estimates of γ. The hard-hexagons values are −0·6667, −0·1111 and 1·444 respectively.

N/D	(i)	(ii)	(iii)
8 9	−0·5849	−0·1098	—
9 8	−0·5800	−0·1139	—
7 9	−0·6053	−0·1104	—
8 8	*	−0·1105	—
9 7	−0·5823	−0·1107	—
7 8	−0·5748	−0·1105	—
8 7	−0·5701	−0·1105	—
6 8	−0·5679	−0·1104	1·400
7 7	−0·5693	−0·1105	1·400
8 6	−0·5692	−0·1101	1·383
6 7	−0·5587	−0·1113	1·402
7 6	−0·5704	−0·1109	1·435
5 7	−0·5077	−0·1097	1·406
6 6	−0·5287	−0·1100	1·409
7 5	−0·6084	−0·1338	1·400
5 6	−0·5522	−0·1090	1·325
6 5	−0·5588	−0·1086	1·415
4 6	−0·5504	−0·1028	1·418
5 5	−0·5504	−0·1071	1·422
6 4	−0·5436	−0·1078	1·419
4 5	−0·5531	−0·1099	1·431
5 4	−0·5338	−0·1090	1·354

Padé approximants to M^{-9} were also constructed. These had poles at $u = 0{\cdot}250\,00 \pm 0{\cdot}000\,03$, confirming that the self-dual point is the critical point and that $\beta = \frac{1}{9}$ as for hard hexagons.

Baxter's exact solution of the 'hard-hexagons' system gives $\alpha' = \frac{1}{3}$, $\beta = \frac{1}{9}$ (and using scaling $\gamma' = 1\frac{4}{9}$). The pattern of estimates in table 2 is similar to that seen by Enting (1980a) when analysing the square-lattice Potts model with two-site interactions. The exponent estimates vary slowly, heading towards the 'hard-hexagons' values as higher-order approximants are considered. Enting (1980a) pointed out that estimates from $[N, N]$ approximants extrapolated linearly against $1/N$ gave reasonable agreement with Baxter's exponents, but even with 31 series terms available the $1/N$ fit was fairly crude.

So far as can be seen from the shorter series, however, the three-site triangular system seems to behave in a manner very similar to the two-site square system. It would seem that not only are the scaling exponents identical, but that the 'corrections-to-scaling' effects are also similar. In each case, analogous techniques of analysis underestimate β (the estimates increase as more terms are considered) and overestimate α (when investigating the energy—see, for example, de Neef and Enting (1977)). Estimates based on the specific heat tend to underestimate α—see Straley and Fisher (1973), Zwansig and Ramshaw (1977) and Enting (1980a). The exponent γ' tends to be underestimated (cf Straley and Fisher 1973).

The correspondence in the critical behaviour of two somewhat different Potts models helps to confirm the existence of a universality class of three-state Potts models, and the trends in the exponents indicate that the 'hard-hexagons' system is, as predicted, a member of this class even though it is, in some ways, a special limiting case. The most surprising prediction that follows is that $\delta = 14$ for the various three-state Potts model. Unfortunately the finite-lattice method is unsuitable for deriving extensive high-field series for square-lattice systems, and even less suitable for triangular systems.

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